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# A non-abelian representation of the dual polar space $DQ(2n, 2)$

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## Abstract

We prove that the dual polar space  $DQ(2n, 2)$ ,  $n \geq 3$ , of rank  $n$  associated with a non-singular parabolic quadric in  $PG(2n, 2)$  admits a faithful non-abelian representation in the extraspecial 2-group  $2_+^{1+2^n}$ . The near  $2n$ -gon  $\mathbb{I}_n$  (section 2.4) is a geometric hyperplane of  $DQ(2n, 2)$ . For  $n \geq 3$ , we first construct a faithful non-abelian representation of  $\mathbb{I}_n$  in  $2_+^{1+2^n}$  and subsequently extend it to a faithful non-abelian representation of  $DQ(2n, 2)$  in  $2_+^{1+2^n}$ .

Keywords: dual polar space, non-abelian representation, extraspecial 2-group

MSC 2000: 05B25

## 1. Introduction

Let  $p$  be a fixed prime number. In [12], Ivanov introduced the notion of representations in groups of point-line geometries  $S = (P, L)$  of order  $p$ , that is, lines of size  $p + 1$ .

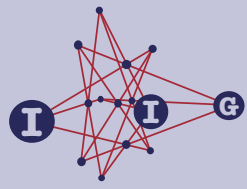
**Definition 1.1** ([12, p. 305]). A representation of  $S$  in a group  $R$  is a mapping  $\psi$  from the point set  $P$  of  $S$  into the set of subgroups of order  $p$  in  $R$  such that the following hold:

- (i)  $R$  is generated by the subgroups  $\psi(x)$ ,  $x \in P$ ;
- (ii) For each line  $l \in L$ , the subgroups  $\psi(x)$ ,  $x \in l$ , are pairwise distinct and generate an elementary abelian  $p$ -subgroup of  $R$  of order  $p^2$ .

This concept of representations in groups of geometries with lines of size  $p + 1$  is similar to the definition of the root group geometries of groups of Lie

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type defined over a field  $\mathbb{F}_p$  with  $p$  elements studied by Cohen and Cooperstein [6, p. 75]. This definition of representations of geometries led to a new research area in the theory of groups and geometries [12]. For example, the knowledge of the representations is crucial for the construction of affine and  $c$ -extensions of geometries and non-split extensions of groups and modules [13, sections 2.7 and 2.8].

We write  $(R, \psi)$  to mean that  $\psi$  is a representation of  $S$  in  $R$ . The group  $R$  is then called a *representation group* of  $S$ . A representation  $(R, \psi)$  of  $S$  is *faithful* if  $\psi$  is injective, is *abelian* or *non-abelian* according as  $R$  is abelian or not. Note that, in [12], ‘non-abelian representation’ means that ‘the corresponding representation group is not necessarily abelian’.

We indicate various possibilities for a representation of a point-line geometry of prime order and the corresponding representation group.

- (1) Every representation of a projective space (as a point-line geometry) is faithful and abelian.
- (2) A representation of a point-line geometry need not be faithful. For example: let  $S = (P, L)$  be a  $(2, 1)$ -GQ and  $R = \{1, r_1, r_2, r_3\}$  be the Klein 4-group. A *triad* of  $S$  is a triple of pairwise non-collinear points of  $S$ . Let  $P_1, P_2, P_3$  be the three triads of  $S$  partitioning the point set  $P$  of  $S$ . Define a map  $\psi$  from  $P$  to the set of subgroups of order 2 in  $R$  by  $\psi(x) = \langle r_i \rangle$  if  $x \in P_i$ . Then  $(R, \psi)$  is an abelian representation of  $S$  which is not faithful.
- (3) The representation group for an abelian representation is an elementary abelian  $p$ -group. So it could be considered as a vector space over  $\mathbb{F}_p$  and the corresponding representation is a full projective embedding which need not be faithful.
- (4) There are point-line geometries, different from the projective spaces, whose representations are always abelian. In [14, Theorems 1.5 and 1.6] it is proved that this is the case for every finite polar space which is not of symplectic type of odd prime order.
- (5) The representation group for a non-abelian representation of a finite point-line geometry could be infinite.

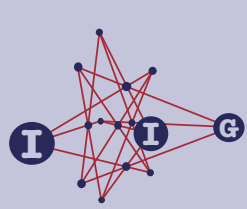
[Let  $S = (P, L)$  be a point-line geometry of order 2 admitting at least one representation. The *universal representation group*  $U(S)$  of  $S$  has the presentation:

$$U(S) = \langle u_x : x \in P, u_x^2 = 1, u_x u_y u_z = 1 \text{ for every } \{x, y, z\} \in L \rangle.$$

Let  $\psi_S$  be the map from  $P$  to the set of subgroups of order 2 in  $U(S)$  defined by  $x \mapsto \langle u_x \rangle$  for  $x \in P$ . Then  $(U(S), \psi_S)$  is a representation

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of  $S$ , called the *universal representation* of  $S$ . Now, let  $S = (P, L)$  be a generalized hexagon with parameters  $(2, 2)$ . Then  $S$  is isomorphic to  $H(2)$  or its dual  $H(2)^*$  [16, Theorem 4, p. 402]. For each  $x \in P$ , consider the geometric hyperplane  $H(x)$  of  $S$  consisting of points at non-maximal distance from  $x$ . The subgraph of the collinearity graph of  $S$  (see section 2 for the definition) induced on the complement of  $H(x)$  in  $P$  is connected if  $S \simeq H(2)$  and has two connected components if  $S \simeq H(2)^*$  [10]. By [12, Lemma 3.6, p. 310], the universal representation group of  $S$  is infinite when  $S \simeq H(2)^*$ .]

In this paper, we prove the following:

**Theorem 1.2.** *Let  $DQ(2n, 2)$ ,  $n \geq 2$ , be the dual polar space of rank  $n$  associated with a non-singular parabolic quadric in  $PG(2n, 2)$ . The following hold:*

- (i) *If  $DQ(2n, 2)$  admits a non-abelian representation, then  $n \geq 3$ .*
- (ii)  *$DQ(2n, 2)$ ,  $n \geq 3$ , admits a faithful non-abelian representation in the extraspecial 2-group  $2_+^{1+2^n}$ .*

## 2. Basic definitions

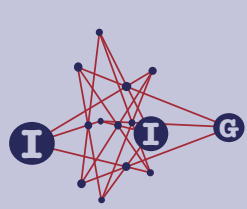
Let  $S = (P, L)$  be a *partial linear space*, that is, a point-line geometry with a ‘point-set’  $P$  and a ‘line set’  $L$  of subsets of  $P$  of size at least two such that any two distinct points of  $S$  are contained in at most one line of  $S$ . If each line of  $S$  contains exactly three points, then  $S$  is called *slim*. For distinct points  $x, y \in P$ , we write  $x \sim y$  if there is a line of  $S$  containing them (we then say that  $x$  and  $y$  are *collinear*). For  $x \in P$  and  $A \subseteq P$ , we define

$$x^\perp = \{x\} \cup \{y \in P : x \sim y\} \text{ and } A^\perp = \bigcap_{x \in A} x^\perp.$$

If  $P^\perp$  is empty, then  $S$  is called *non-degenerate*. The graph  $\Gamma(P)$  with vertex set  $P$ , in which two distinct vertices are adjacent whenever they are collinear in  $S$ , is called the *collinearity graph* of  $S$ . If  $\Gamma(P)$  is connected, then  $S$  is a *connected* partial linear space. A subset  $X$  of  $P$  is a *subspace* of  $S$  if any line of  $S$  containing at least two points of  $X$  is entirely contained in  $X$ . A subspace  $X$  of  $S$  is *singular* if  $x \sim y$  for every pair of distinct points  $x, y \in X$ , that is, the induced subgraph  $\Gamma(X)$  of  $\Gamma(P)$  is a clique. A *geometric hyperplane* of  $S$  is a subspace of  $S$  different from  $P$ , that meets each line of  $S$  non-trivially. Two partial linear spaces  $S = (P, L)$  and  $S' = (P', L')$  are *isomorphic*, written as  $S \simeq S'$ , if there exists a bijection  $\alpha : P \rightarrow P'$  such that  $\alpha(x) \sim \alpha(y)$  in  $S'$  whenever  $x \sim y$  in  $S$  and it induces a bijection from  $L$  to  $L'$ . Such a map  $\alpha$  is called an *isomorphism* from  $S$  to  $S'$ .

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## 2.1. Near polygons

A *near polygon* [15] is a partial linear space  $S = (P, L)$  of finite diameter (that is, the diameter of  $\Gamma(P)$  is finite) such that the following ‘near polygon’ property holds:

*For each point-line pair  $(x, l) \in P \times L$ , there exists a unique point in  $l$  which is nearest to  $x$ .*

Here, the distance  $d(x, y)$  between two points  $x$  and  $y$  of  $S$  is measured in the graph  $\Gamma(P)$ . If the diameter of  $S$  is  $n$ , then the near polygon  $S$  is called a *near  $2n$ -gon*. For  $x \in P$ , we define

$$\begin{aligned}\Gamma_n(x) &= \{y \in P \mid d(x, y) = n\}; \\ \Gamma_{<n}(x) &= \{y \in P \mid d(x, y) < n\}.\end{aligned}$$

For every  $x \in P$  with  $\Gamma_n(x) \neq \emptyset$ ,  $\Gamma_{<n}(x)$  is a geometric hyperplane of  $S$ . If  $n = 2$  and  $S$  is non-degenerate, then  $S$  is a *generalized quadrangle* (GQ, for short). If a finite generalized quadrangle has a line containing at least three points and a point contained in at least three lines, then there exist integers  $s$  and  $t$  such that each line contains  $s + 1$  points and each point is contained in  $t + 1$  lines [3, Theorem 7.1, p. 98]. In that case, we say that it is an  $(s, t)$ -GQ.

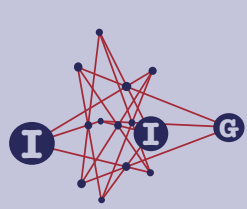
Let  $S = (P, L)$  be a near polygon. If every line of  $S$  contains at least three points and if every two points of  $S$  at distance 2 have at least two common neighbours, then  $S$  is called a *dense near polygon*. A subspace  $C$  of  $S$  is *convex* if every geodesic in  $\Gamma(P)$  between two points of  $C$  is entirely contained in  $C$ . A *quad* is a convex subspace of  $S$  of diameter 2 such that no point of it is adjacent to all other points of it. The points and the lines contained in a quad define a generalized quadrangle. If  $x$  and  $y$  are two points of a dense near polygon at distance 2 from each other, then there is a unique quad containing  $x$  and  $y$  [15, Proposition 2.5, p. 10].

Let  $S = (P, L)$  be a slim dense near  $2n$ -gon. If  $n = 1$ , then  $S \simeq \mathbb{L}_3$ , a line of size 3. If  $n = 2$ , then  $S$  is a  $(2, t)$ -GQ. In that case,  $P$  is finite,  $t = 1, 2$  or  $4$  and for each such value of  $t$  there exists a unique  $(2, t)$ -GQ, up to isomorphism [3, Theorem 7.3, p. 99]. Thus,  $S$  is isomorphic to one of the classical generalized quadrangles  $Q^+(3, 2)$ ,  $W(2) \simeq Q(4, 2)$  and  $Q^-(5, 2)$  for  $t = 1, 2$  and  $4$ , respectively. We refer to [7] for the classification of all slim dense near  $2n$ -gons when  $n \in \{3, 4\}$ .

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## 2.2. Dual polar spaces

Here, a *polar space* is a non-degenerate point-line geometry  $S = (P, L)$  satisfying the following ‘one or all’ axiom (see [2, Theorem 4, p. 161] and [17, 7.1, p. 102]):

*For each point-line pair  $(x, l) \in P \times L$  with  $x \notin l$ ,  $x$  is collinear with one or all points of  $l$ .*

A polar space is a partial linear space [2, Theorem 3]. The *rank* of a polar space  $S$  is the supremum of the lengths  $m$  of chains  $Q_0 \subsetneq Q_1 \subsetneq \dots \subsetneq Q_m$  of singular subspaces in  $S$ . A polar space of rank 2 is a generalized quadrangle.

Let  $S = (P, L)$  be a finite polar space of rank  $n \geq 2$ . Every singular subspace of  $S$  is isomorphic to a projective space. The *dimension* of a singular subspace of  $S$  is the dimension of the associated projective space. Each maximal singular subspace of  $S$  has dimension  $n - 1$  [2, Proposition 11]. For singular subspaces  $X$  and  $Y$  of  $S$  with  $Y \subset X$ , the *co-dimension* of  $Y$  in  $X$  is the dimension of  $X$  minus the dimension of  $Y$ . Consider the point-line geometry  $DS = (P', L')$ , where

- $P'$  is the collection of all maximal singular subspaces of  $S$ ;
- an element of  $L'$  is the collection of all maximal singular subspaces of  $S$  containing a specific singular subspace of  $S$  of co-dimension 1 in each of them.

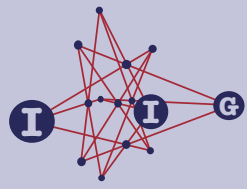
Then  $DS$  is a partial linear space, called the *dual polar space of rank  $n$*  associated with  $S$ . Cameron characterized these geometries in terms of points and lines and proved that dual polar spaces of rank  $n$  are dense near  $2n$ -gons [4, Theorem 1, p. 75].

## 2.3. The dual polar space $DQ(2n, 2)$

Let  $Q(2n, 2)$ ,  $n \geq 2$ , be a non-singular parabolic quadric in  $PG(2n, 2)$ . Then the points together with the lines of  $Q(2n, 2)$  is a polar space of rank  $n$  and  $DQ(2n, 2)$  is the associated dual polar space of rank  $n$ . Thus, the points of  $DQ(2n, 2)$  are the generators (that is, subspaces of maximal dimension  $n - 1$ ) of  $Q(2n, 2)$  and a line of  $DQ(2n, 2)$  is a triple of generators containing a specific  $(n - 2)$ -dimensional subspace of  $Q(2n, 2)$ . The dual polar space  $DQ(2n, 2)$  is a slim dense near  $2n$ -gon. The quads of  $DQ(2n, 2)$  are isomorphic to  $W(2)$ , the unique  $(2, 2)$ -GQ. For each point  $x$  of  $DQ(2n, 2)$ , the set  $\Gamma_{<n}(x)$  is a maximal subspace of  $DQ(2n, 2)$ .

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## 2.4. The near $2n$ -gon $\mathbb{I}_n$

Again, consider a non-singular parabolic quadric  $Q(2n, 2)$ ,  $n \geq 2$ , in  $\text{PG}(2n, 2)$  and a hyperplane of  $\text{PG}(2n, 2)$  which intersects  $Q(2n, 2)$  in a non-singular hyperbolic quadric  $Q^+(2n - 1, 2)$ . The set  $X$  of all generators of  $Q(2n, 2)$  which are not contained in  $Q^+(2n - 1, 2)$  is a subspace of  $DQ(2n, 2)$  [7, Theorem 6.46, p. 140]. The points and the lines of  $DQ(2n, 2)$  contained in  $X$  define a slim dense near  $2n$ -gon [7, Theorem 6.48, p. 141], denoted by  $\mathbb{I}_n$ . For  $n = 2$ , the generalized quadrangle  $\mathbb{I}_2$  is isomorphic to  $Q^+(3, 2)$ . For  $n \geq 3$ , each quad of  $\mathbb{I}_n$  is either a  $(2, 1)$ -GQ or a  $(2, 2)$ -GQ.

## 2.5. Extraspecial 2-groups

A finite 2-group  $G$  is *extraspecial* if the Frattini subgroup  $\Phi(G)$ , the commutator subgroup  $G' = [G, G]$  and the center  $Z(G)$  of  $G$  coincide and have order 2. We refer to [9, section 20, p. 78,79] —see also [11, chapter 5, section 5]— for the following properties of an extraspecial 2-group.

An extraspecial 2-group is of order  $2^{1+2m}$  for some integer  $m \geq 1$ . Let  $D_8$  and  $Q_8$ , respectively, denote the *dihedral* and *quaternion groups* of order 8. A non-abelian 2-group of order 8 is extraspecial and is isomorphic to  $D_8$  or  $Q_8$ . Let  $G$  be an extraspecial 2-group of order  $2^{1+2m}$ . Then the exponent of  $G$  is 4 and either

- (i)  $G$  is a central product of  $m$  copies of  $D_8$ , or
- (ii)  $G$  is a central product of  $m - 1$  copies of  $D_8$  and a copy of  $Q_8$ .

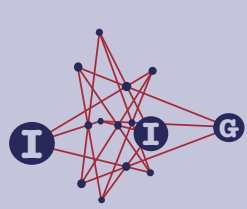
So, the maximum of the orders of its abelian subgroups is  $2^{m+1}$ . In case (i),  $G$  possesses a maximal abelian subgroup of order  $2^{m+1}$  which is elementary abelian. In case (ii), each maximal abelian subgroup of  $G$  is isomorphic to  $C_2^{m-1} \times C_4$ . Here,  $C_k$  denotes the cyclic group of order  $k$ . We denote an extraspecial 2-group of order  $2^{1+2m}$  by  $2_+^{1+2m}$  if (i) holds, and by  $2_-^{1+2m}$  if (ii) holds.

## 3. Proof of Theorem 1.2

Let  $S = (P, L)$  be a slim partial linear space and  $(R, \psi)$  be a representation of  $S$ . For each  $x \in P$ , we identify the subgroup  $\psi(x) = \langle r_x \rangle$  of  $R$  with its non-trivial element  $r_x$ . If  $x, y \in P$  and  $x \sim y$ , then we denote by  $xy$  the unique line of  $S$  containing  $x$  and  $y$ , and define  $x * y$  by  $xy = \{x, y, x * y\}$ . So,  $r_{x*y} = r_x r_y$  for every line  $\{x, y, x * y\}$  of  $S$ , by condition (ii) of Definition 1.1. The following

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lemma is a particular case of [14, Proposition 3.1, p. 59]. We write down the proof here for the sake of completeness of this paper. (In the statement of [14, Proposition 3.1, p. 59], the polar space should be of order 2.)

**Lemma 3.1.** *Let  $S = (P, L)$  be a  $(2, t)$ -GQ and  $(R, \psi)$  be a representation of  $S$ . Then  $R$  is abelian.*

*Proof.* We show that  $[r_x, r_y] = 1$  for all  $x, y \in P$  with  $x \approx y$ . Let  $Q$  be a  $(2, 1)$ -subGQ in  $S$  containing  $x$  and  $y$ . Such a  $Q$  exists, follows from the fact that each line contains exactly 3 points. Let  $\{x, y\}^\perp \cap Q = \{a, b\}$ . In  $Q$ ,  $[r_b, r_y] = [r_b, r_x] = 1$  and  $r_{(a*x)*(b*y)} = r_{(a*y)*(b*x)}$ , implies that  $r_x r_y = r_y r_x$ .  $\square$

The dual polar space  $DQ(4, 2)$  is a  $(2, 2)$ -GQ which is isomorphic to  $W(2)$ . By Lemma 3.1, every representation of it is abelian. This proves Theorem 1.2(i).

We next construct non-abelian representations for the near  $2(n+1)$ -gon  $\mathbb{I}_{n+1}$  and the dual polar space  $DQ(2n+2, 2)$  in the group  $2_+^{1+2 \cdot 2^n}$  when  $n \geq 2$ . We make use of the following recursive constructions of  $\mathbb{I}_{n+1}$  and  $DQ(2n+2, 2)$  given by De Bruyn [8].

Let  $\mathbb{S}_n = (\mathbb{P}_n, \mathbb{L}_n)$  denote the dual polar space  $DQ(2n, 2)$  of rank  $n \geq 2$ . The quads of  $\mathbb{S}_n$  are  $(2, 2)$ -GQs. Every triad  $\{a, b, c\}$  of points contained in a quad of  $\mathbb{S}_n$  has the property that  $\{a, b, c\}^\perp$  contains one or three points. In the latter case, such a triad  $\{a, b, c\}$  is called a *hyperbolic line* of  $\mathbb{S}_n$ . Thus,  $\{a, b, c\}$  is a hyperbolic line of  $\mathbb{S}_n$  if and only if  $\{a, b, c\}^\perp$  is so. Now, consider the point-line geometries  $\mathcal{S}_{n+1} = (\mathcal{P}_{n+1}, \mathcal{L}_{n+1})$  and  $\mathbb{S}_{n+1} = (\mathbb{P}_{n+1}, \mathbb{L}_{n+1})$  constructed from  $\mathbb{S}_n$ , where

$$\mathcal{P}_{n+1} = \{(x, y) \in \mathbb{P}_n \times \mathbb{P}_n : y \in x^\perp\};$$

$$\mathcal{L}_{n+1} = \{ \{(x, u), (y, v), (z, w)\} : \{x, y, z\} \text{ is a line or a hyperbolic line of } \mathbb{S}_n \\ \text{and } \{x, y, z\}^\perp = \{u, v, w\} \};$$

$$\mathbb{P}_{n+1} = \mathcal{P}_{n+1} \cup \mathbb{P}_n \cup \bar{\mathbb{P}}_n, \text{ where } \bar{\mathbb{P}}_n = \{\bar{x} : x \in \mathbb{P}_n\};$$

$$\mathbb{L}_{n+1} = \mathcal{L}_{n+1} \cup \mathcal{L}^1, \text{ where } \mathcal{L}^1 = \{ \{x, (x, u), \bar{u}\} : (x, u) \in \mathcal{P}_{n+1} \}.$$

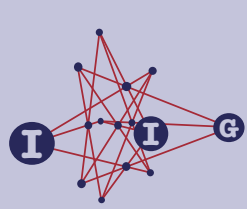
Then  $\mathcal{S}_{n+1}$  is isomorphic to the near  $2(n+1)$ -gon  $\mathbb{I}_{n+1}$  and  $\mathbb{S}_{n+1}$  is isomorphic to the dual polar space  $DQ(2n+2, 2)$  [8, section 1.5, Corollary 1.3 and Theorem 1.4].

Now, let  $R = 2_+^{1+2 \cdot 2^n}$ ,  $n \geq 2$ . The quotient group  $R/R'$  is an elementary abelian 2-group. Set  $R' = \langle \theta \rangle$  and  $V = R/R'$ . Consider  $V$  as a vector space of dimension  $2^{n+1}$  over  $\mathbb{F}_2$ . The map

$$f: V \times V \rightarrow \mathbb{F}_2$$

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defined by

$$f(xR', yR') = i,$$

where  $[x, y] = \theta^i$ ,  $i \in \{0, 1\}$ , is a non-degenerate symplectic bilinear form on  $V$  [9, Theorem 20.4, p. 78]. We write  $V$  as an orthogonal direct sum of  $2^n$  hyperbolic planes  $K_i$ ,  $1 \leq i \leq 2^n$ , in  $V$  with respect to  $f$ . Let  $H_i$  be the inverse image of  $K_i$  in  $R$  under the natural surjective homomorphism from  $R$  to  $V$ . Then  $H_i$  is generated by two elements  $x_i$  and  $y_i$  such that  $[x_i, y_i] = \theta$ . We set

$$M = \langle x_i : 1 \leq i \leq 2^n \rangle; \quad \bar{M} = \langle y_i : 1 \leq i \leq 2^n \rangle.$$

Then  $M$  and  $\bar{M}$  are elementary abelian 2-subgroups of  $R$  each of order  $2^{2^n}$ . The groups  $M$ ,  $\bar{M}$  and  $R'$  pairwise intersect trivially and  $R = M\bar{M}R'$ . Further,  $C_{\bar{M}}(M)$  and  $C_M(\bar{M})$  are trivial.

Let  $(M, \tau)$  be the faithful abelian representation of  $\mathbb{S}_n$  arising from the spin-embedding of  $DQ(2n, 2)$  in a vector space of dimension  $2^n$ . We refer to [1] for a description of the spin-embedding. Then the following property  $(\star)$  is satisfied:

$(\star)$  For every point  $x$  of  $\mathbb{S}_n$ , the subgroup  $\langle m_y : y \in \Gamma_{<n}(x) \rangle$  is of index 2 in  $M$ .

This embedding of  $DQ(2n, 2)$  is the so-called minimal full polarized embedding of  $DQ(2n, 2)$  in the sense of [5] and the property  $(\star)$  is the condition of polarization for a projective full embedding.

Let  $Q$  be a quad in  $DQ(2n, 2)$ . Let  $G = \langle \tau(Q) \rangle$ . Then  $(G, \tau)$  is a faithful abelian representation of  $Q$ . Since  $Q$  is a  $(2, 2)$ -GQ,  $G$  is of order  $2^4$  or  $2^5$ . Since  $(M, \tau)$  is minimal and polarized and  $Q$  is a convex subspace of  $DQ(2n, 2)$ , it follows from [5, Theorem 1.6, p. 10] that  $(G, \tau)$  is also minimal and polarized. This implies that  $G$  is of order  $2^4$ .

**Lemma 3.2.** Let  $a, b, c$  be three pairwise distinct points of  $Q$ . Then  $T = \{a, b, c\}$  is a line or a hyperbolic line of  $Q$  if and only if  $g_a g_b g_c = 1$ .

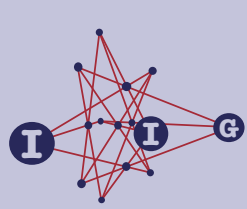
*Proof.* First, assume that  $T$  is a hyperbolic line of  $Q$ . Let  $Q'$  be a  $(2, 1)$ -subGQ of  $Q$  containing  $a$  and  $b$ . Then  $c \notin Q'$  and  $Q = \langle Q', c \rangle$ . Let  $\{x, y\} = \{a, b\}^\perp \cap Q'$ . Then  $x, y \in T^\perp$ , since  $T$  is a hyperbolic line. Let  $z$  be the unique point in  $Q'$  such that  $\{x, y, z\}$  is a triad of  $Q'$ . Then  $c \sim z$  and  $g_z = g_{a*x} g_{b*y} = (g_a g_x)(g_b g_y)$ . Since the subgroup  $H = \langle g_y : y \in x^\perp \cap Q \rangle$  is of index 2 in  $G$ ,  $|H| = 2^3$  and  $H = \langle g_x, g_a, g_b \rangle$ . So  $g_c$  is equal to either  $g_a g_b$  or  $g_a g_b g_x$ , since  $\tau$  is faithful. If the latter holds, then  $g_{c*z} = g_c g_z = g_y$ . But this is not possible, since  $y \neq c * z$  and  $\tau$  is faithful. Thus  $g_c = g_a g_b$  and so  $g_a g_b g_c = 1$ .

Now assume that  $g_a g_b g_c = 1$  and that  $T$  is not a line. Then  $T$  is a triad, since  $\tau$  is faithful. We show that  $T$  is a hyperbolic line. Suppose that  $T$  is not

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a hyperbolic line. Then  $|T^\perp| = 1$ . Let  $\{a, b\}^{\perp\perp} = \{a, b, d\}$ . Since  $\{a, b, d\}$  is a hyperbolic line,  $g_a g_b g_d = 1$  by the first part. Since  $|T^\perp| = 1$ ,  $c \neq d$  and  $g_c = g_a g_b = g_d$ , a contradiction to that  $\tau$  is faithful. Hence  $T$  is a hyperbolic line of  $Q$ .  $\square$

For each point  $x$  of  $\mathbb{S}_n$ , set  $H_x = \langle m_y : y \in \Gamma_{<n}(x) \rangle$ . Since  $H_x$  is a maximal subgroup of  $M$ , the centralizer of  $H_x$  in  $\bar{M}$  is a subgroup  $\langle \bar{m}_x \rangle$  of order 2. Since  $\Gamma_{<n}(x)$  is a maximal subspace of  $\mathbb{S}_n$ ,  $\mathbb{P}_n = \langle \Gamma_{<n}(x) \cup \{w\} \rangle$  and  $M = \langle H_x, m_w \rangle$  for  $w \in \Gamma_n(x)$ . The triviality of  $C_{\bar{M}}(M)$  implies that  $[\bar{m}_x, m_w] = \theta$  for every  $w \in \Gamma_n(x)$ .

Recall that  $\bar{\mathbb{P}}_n = \{\bar{x} : x \in \mathbb{P}_n\}$ . Let  $\bar{\mathbb{L}}_n = \{\{\bar{x}, \bar{y}, \bar{z}\} : \{x, y, z\} \in \mathbb{L}_n\}$ . Then  $\bar{\mathbb{S}}_n = (\bar{\mathbb{P}}_n, \bar{\mathbb{L}}_n) \simeq DQ(2n, 2)$ . Let  $\bar{\tau}$  be the map from the point set  $\bar{\mathbb{P}}_n$  of  $\bar{\mathbb{S}}_n$  to  $\bar{M}$  defined by  $\bar{\tau}(\bar{x}) = \bar{m}_x$ .

**Proposition 3.3.**  $(\bar{M}, \bar{\tau})$  is a faithful abelian representation of  $\bar{\mathbb{S}}_n$  satisfying the property  $(\star)$ .

*Proof.* For  $x \neq y$  in  $\mathbb{P}_n$ ,  $\Gamma_{<n}(x) \neq \Gamma_{<n}(y)$ . So  $H_x \neq H_y$  and  $C_{\bar{M}}(H_x) \neq C_{\bar{M}}(H_y)$ . This implies that  $\bar{m}_x \neq \bar{m}_y$  and hence  $\bar{\tau}$  is injective.

Let  $\{x, y, z\}$  be a line of  $\mathbb{S}_n$ . Let  $w \in \Gamma_{<n}(z)$ . Then  $d(w, x) \leq n - 1$  if and only if  $d(w, y) \leq n - 1$ , by the ‘near polygon’ property. So  $([\bar{m}_x, m_w], [\bar{m}_y, m_w])$  is equal to either  $(1, 1)$  or  $(\theta, \theta)$ . Then

$$[\bar{m}_x \bar{m}_y, m_w] = [\bar{m}_x, m_w][\bar{m}_y, m_w] = 1.$$

The first equality holds, since  $R$  has nilpotent class 2. Thus,  $1 \neq \bar{m}_x \bar{m}_y \in C_{\bar{M}}(H_z)$ . Since  $\bar{m}_z$  is the unique non-trivial element in  $C_{\bar{M}}(H_z)$ , it follows that  $\bar{m}_z = \bar{m}_x \bar{m}_y$ . So,  $\bar{m}_x \bar{m}_y \bar{m}_z = 1$  for every line  $\{\bar{x}, \bar{y}, \bar{z}\}$  of  $\bar{\mathbb{S}}_n$ . This verifies condition (ii) of Definition 1.1.

Now, let  $K = \langle \bar{\tau}(\bar{\mathbb{P}}_n) \rangle$ . Then  $(K, \bar{\tau})$  is a faithful abelian representation of  $\bar{\mathbb{S}}_n$ . For each  $\bar{x} \in \bar{\mathbb{P}}_n$ ,  $H_{\bar{x}} = \langle \bar{m}_y : \bar{y} \in \Gamma_{<n}(\bar{x}) \rangle$  is equal to  $K$  or is of index 2 in  $K$ . Since  $m_x$  commutes with each element of  $H_{\bar{x}}$  and  $m_x$  does not commute with  $m_w$  for  $w \in \Gamma_n(x)$ , the first possibility does not occur. This implies that the property  $(\star)$  holds.

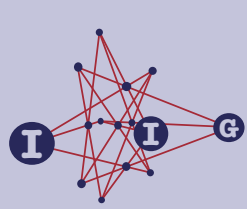
Since  $\bar{\mathbb{S}}_n \simeq DQ(2n, 2)$  does not possess a faithful polarized projective embedding in a vector space of dimension less than  $2^n$  [5], it follows that  $K = \bar{M}$ . So condition (i) of Definition 1.1 holds, thus completing the proof.  $\square$

By Proposition 3.3, a similar statement in Lemma 3.2 holds for the restriction of  $\bar{\tau}$  to a quad of  $\bar{\mathbb{S}}_n$ . Now, let  $\beta: \mathcal{P}_{n+1} \rightarrow R$  be defined by

$$\beta((x, y)) = m_x \bar{m}_y,$$

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for  $(x, y) \in \mathcal{P}_{n+1}$ . Since  $[m_x, \bar{m}_y] = 1$  for  $x, y \in \mathbb{P}_n$  with  $y \in x^\perp$ ,  $\beta((x, y)) = m_x \bar{m}_y$  is of order 2 in  $R$  for every point  $(x, y)$  of  $\mathcal{S}_{n+1}$ .

**Proposition 3.4.**  $(R, \beta)$  is a faithful non-abelian representation of  $\mathcal{S}_{n+1} \simeq \mathbb{I}_{n+1}$ .

*Proof.* If  $\beta((x, u)) = \beta((y, v))$ , then  $m_x \bar{m}_u = m_y \bar{m}_v$  implies that  $m_y m_x = \bar{m}_v \bar{m}_u$ . Since  $M \cap \bar{M}$  is trivial, it follows that  $m_x = m_y$  and  $\bar{m}_u = \bar{m}_v$ . This implies that  $\beta$  is one-one.

We now verify conditions (i) and (ii) of Definition 1.1. Let  $x \in \mathbb{P}_n$ . Let  $\{x, y, z\}$  be a hyperbolic line of  $\mathbb{S}_n$  containing  $x$  and let  $u \in \{x, y, z\}^\perp$ . Then

$$\beta((y, u)) \beta((z, u)) = m_y \bar{m}_u m_z \bar{m}_u = m_y m_z = m_x.$$

The last equality follows from Lemma 3.2. Thus,  $m_x \in \langle \beta(\mathcal{P}_{n+1}) \rangle$  for every  $x \in \mathbb{P}_n$ . This also implies that  $\bar{m}_x \in \langle \beta(\mathcal{P}_{n+1}) \rangle$  for every  $x \in \mathbb{P}_n$ . In particular,  $M$  and  $\bar{M}$  are contained in  $\langle \beta(\mathcal{P}_{n+1}) \rangle$ . Since  $R$  is generated by  $M$  and  $\bar{M}$ , we get  $R = \langle \beta(\mathcal{P}_{n+1}) \rangle$ . Now, let  $\{(x, u), (y, v), (z, w)\}$  be a line of  $\mathcal{S}_{n+1}$ . We have

$$\beta((x, u)) \beta((y, v)) = (m_x \bar{m}_u)(m_y \bar{m}_v) = m_x m_y \bar{m}_u \bar{m}_v r' = m_z \bar{m}_w r',$$

where  $r' = [\bar{m}_u, m_y]$ . The last equality holds by Lemma 3.2, since  $\{x, y, z\}$  and  $\{\bar{u}, \bar{v}, \bar{w}\}$  are lines or hyperbolic lines of  $\mathbb{S}_n$  and  $\bar{\mathbb{S}}_n$  respectively. Since  $y \in u^\perp$  in  $\mathbb{S}_n$ , we get  $r' = 1$ . So,  $\beta((x, u)) \beta((y, v)) = m_z \bar{m}_w = \beta((z, w))$ .  $\square$

*Proof of Theorem 1.2.* Let  $R, M, \bar{M}, \tau, \bar{\tau}$  and  $\beta$  be as in the above. Let  $\rho$  be the map from  $\mathbb{P}_{n+1}$  to  $R$  defined by

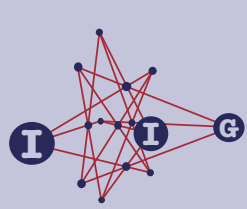
$$\rho = \begin{cases} \tau & \text{on } \mathbb{P}_n; \\ \bar{\tau} & \text{on } \bar{\mathbb{P}}_n; \\ \beta & \text{on } \mathcal{P}_{n+1}. \end{cases}$$

Then  $R = \langle \rho(\mathbb{P}_{n+1}) \rangle$ . Also, condition (ii) of Definition 1.1 holds for every line in  $\mathcal{L}^1$ . As a consequence of Proposition 3.4,  $(R, \rho)$  is a faithful non-abelian representation of  $\mathbb{S}_{n+1} \simeq DQ(2n+2, 2)$ . This completes the proof.  $\square$

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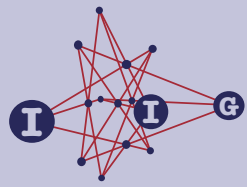
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## References

- [1] **F. Buekenhout** and **P. J. Cameron**, Projective and affine geometry over division rings, in *Handbook of incidence geometry*, North-Holland, Amsterdam, 1995, 27–62.
- [2] **F. Buekenhout** and **E. E. Shult**, On the foundations of polar geometry, *Geom. Dedicata* **3** (1974), 155–170.
- [3] **P. J. Cameron**, *Projective and polar spaces*, available from <http://www.maths.qmul.ac.uk/~pjc/pps/>.
- [4] ———, Dual polar spaces, *Geom. Dedicata* **12** (1982), 75–85.
- [5] **I. Cardinali**, **B. De Bruyn** and **A. Pasini**, Minimal full polarized embeddings of dual polar spaces, *J. Algebraic Combin.* **25** (2007), 7–23.
- [6] **A. M. Cohen** and **B. N. Cooperstein**, A characterization of some geometries of Lie type, *Geom. Dedicata* **15** (1983), 73–105.
- [7] **B. De Bruyn**, *Near polygons*, Birkhäuser Verlag, Basel, Front. Math., 2006.
- [8] ———, A recursive construction for the dual polar spaces  $DQ(2n, 2)$ , *Discrete Math.* **308** (2008), 5504–5515.
- [9] **K. Doerk** and **T. Hawkes**, *Finite soluble groups*, de Gruyter Exp. Math., vol. 4, Walter de Gruyter & Co., Berlin, 1992.
- [10] **D. Frohardt** and **P. M. Johnson**, Geometric hyperplanes in generalized hexagons of order  $(2, 2)$ , *Comm. Algebra* **22** (1994), 773–797.
- [11] **D. Gorenstein**, *Finite groups*, Chelsea Publishing Co., New York, 1980.
- [12] **A. A. Ivanov**, Non-abelian representations of geometries, *Groups and combinatorics – in memory of Michio Suzuki*, *Adv. Stud. Pure Math.* **32** (2001), 301–314.
- [13] **A. A. Ivanov** and **S. V. Shpectorov**, *Geometry of sporadic groups II – representations and amalgams*, Encyclopedia Math. Appl., vol 91, Cambridge University Press, Cambridge, 2002.
- [14] **B. K. Sahoo** and **N. S. N. Sastry**, A characterization of finite symplectic polar spaces of odd prime order, *J. Combin. Theory Ser. A* **114** (2007), 52–64.
- [15] **E. E. Shult** and **A. Yanushka**, Near  $n$ -gons and line systems, *Geom. Dedicata* **9** (1980), 1–72.

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quit

[16] **J. A. Thas**, Generalized polygons, **in** *Handbook of incidence geometry*, North-Holland, Amsterdam, 1995, 383–431.

[17] **J. Tits**, *Buildings of spherical type and finite BN-pairs*, Lecture Notes Math. **386** (1974), Springer-Verlag, Berlin, New York.

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